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A GROUP PURSUIT PROBLEM WITH PHASE CONSTRAINTS*

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We shall consider the problem of the pursuit of a group of controlled objects by a group of controlled objects, with phase constraints on the states of the evaders.

1. In B R^k $(k \geqslant 2)$, we consider a differential game with $n+m$ players: n pursuers P_i, $i = 1, 2, \ldots, n$, and m evaders E_j, $j = 1, 2, \ldots, m$.

The law of motion of each of the pursuers P_i is

$$x_i^{\cdot} = ax_i + u_i, \quad \|u_i\| \leqslant 1, \quad a < 0 \tag{1.1}$$

The law of motion of each of the evaders E_j is

$$y_j^{\cdot} = ay_j + v, \quad \|v\| \leqslant 1 \tag{1.2}$$

At $t = 0$ the initial positions of the pursuers $x_1^\circ, \ldots, x_n^\circ$ and of the evaders $y_1^\circ, \ldots, y_m^\circ$ are given, and it is assumed that

$$x_i^\circ \neq y_j^\circ, \quad \forall i, \forall j$$

Let $z^\circ = (x_1^\circ, \ldots, x_n^\circ, y_1^\circ, \ldots, y_m^\circ)$. We shall assume that during the game the evaders never leave a convex polyhedral set

$$D = \{z \mid z \in R^k, \ \langle p_\lambda, z\rangle \leqslant \mu_\lambda, \ \lambda = 1, \ldots, r\}$$

where $p_1, \ldots, p_r$ are unit vectors, such that $\operatorname{Int} D \neq \varnothing$. Let $T > 0$ be an arbitrary number and σ some finite partition $t_0 = 0 < t_1 < \ldots < t_{s+1} = T$ of the interval $[0, T]$.

Definition 1. A piecewise-programmed strategy V_j for player E_j, defined on $[0, T]$, corresponding to the partition σ, is defined to be a family of mappings $b^l{}_j$, $l = 0, 1, \ldots, s$, that associate with the quantities

$$(t_l, x_1(t_l), \ldots, x_n(t_l), y_1(t_l), \ldots, y_m(t_l)) \tag{1.3}$$

a measurable function $v_l^j(t)$ defined for $t \in [t_l, t_{l+1})$, such that $\|v_l^j(t)\| \leqslant 1$, $y_j(t) \in D$, $t \in [t_l, t_{l+1})$.

Definition 2. A piecewise-programmed counter-strategy U_j for player P_j, corresponding to the partition σ, is defined to be a family of mappings c_j^l, $l = 0, 1, \ldots, s$, that associate with the quantities (1.3) and the controls $v_l^\mu(t)$, $t \in [t_l, t_{l+1})$, $\mu = 1, 2, \ldots, m$ a measurable function $u_l^j(t)$ defined for $t \in [t_l, t_{l+1})$, such that $\|u_l^j(t)\| \leqslant 1$, $t \in [t_l, t_{l+1})$.

Denote the game by $\Gamma = \Gamma(n, m, z^\circ, D)$.

Definition 3. We shall say that encounter avoidance is possible in Γ if, for any $T > 0$, there exist a partition σ of the interval $[0, T]$ and a strategy V_j for each player E_j, corresponding to σ, such that for any trajectories $x_i(t)$ of players P_i there exists a number $p \in \{1, 2, \ldots, m\}$ such that

$$y_p(t) \neq x_i(t), \quad t \in [0, T]$$

where $y_p(t)$ is the trajectory of E_p realized in the given situation.

Definition 4. We shall say that capture is possible in Γ if there exists $T > 0$ such that, for any trajectories $y_j(t)$ of players E_j and any partition σ of the interval $[0, T]$, there exist piecewise-programmed counter-strategies U_i of players P_i corresponding to the partition σ, times $\tau_j \in [0, T]$ and numbers $p_j \in \{1, 2, \ldots, n\}$, such that

$$y_j(\tau_j) = x_{p_j}(\tau_j),$$

where $x_{p_j}(t)$ is the trajectory of player P_{p_j} realized in the given situation.

2. Consider the game $\Gamma_1 = \Gamma(n, 1, z^\circ, D)$. We may assume that $n \geqslant k$, since if $n < k$ one can show, using results of /1, 2/, that encounter avoidance is possible in this game.

Definition 5. We shall say that vectors a_l, $l = 1, \ldots, s$, form a positive basis of R^k if, for any $x \in R^k$, there exist $\alpha_l > 0$, $l = 1, \ldots, s$, such that

$$x = \alpha_1 a_1 + \ldots + \alpha_s a_s$$

Assuming that $a_1, \ldots, a_s$ are unit vectors, let us consider the functions

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$$\rho_l(a_l, v) = \langle a_l, v\rangle + \sqrt{\langle a_l, v\rangle^2 + 1 - \|v\|^2}$$
$$l = 1, 2, \ldots, s_1, \quad s_1 \leqslant s, \quad s_1 \geqslant 1 \tag{2.1}$$
$$\rho_l(a_l, v) = \langle a_l, v\rangle, \quad l = s_1 + 1, \ldots, s$$
$$\|v\| \leqslant 1$$

Then:

*Theorem 1.**(*Petrov N.N., Simple pursuit when there are phase constraints. (Preprint), Leningrad, Deposited at VINITI, 27.03.84, 1684-84, 1984.) The vectors $a_1, \ldots, a_s$ form a positive basis of R^k if and only if

$$\delta = \min_{\|v\| \leqslant 1} \max_l \rho(a_l, v) > 0$$

Remark 1. If $a_1, \ldots, a_s$ form a positive basis, then $s \geqslant k + 1$.

Instead of systems (1.1), (1.2), we shall consider the system

$$z_i^{\cdot} = az_i + u_i - v, \quad z_i^\circ = x_i^\circ - y^\circ \tag{2.2}$$

Define vectors $a_1, \ldots, a_{r+n}$ as follows:

$$a_i = z_i^\circ / \|z_i^\circ\| \tag{2.3}$$
$$a_{n+l} = p_l, \quad l = 1, 2, \ldots, r$$

Theorem 2. Capture is possible in Γ_1 if and only if the vectors (2.3) form a positive basis.

Proof. Suppose that the vectors (2.3) do not form a positive basis. We shall show that encounter avoidance is possible in Γ_1.

Take a vector $p^\circ, \|p^\circ\| = 1$, such that $\langle a_j, p^\circ\rangle \leqslant 0$. Define a strategy V as follows:

$$\sigma = \{0, +\infty\}, \quad v(t) = p^\circ, \forall t \geqslant 0$$

It is obvious that V is an admissible strategy. It follows from /1/ that

$$z_i(t) \neq 0, \quad \forall t \geqslant 0 \tag{2.4}$$

Hence encounter avoidance will occur in Γ_1.

Now let the vectors (2.3) form a positive basis. We shall show that capture is possible in Γ_1.

The proof proceeds as follows.

1. If $r = 0$, Theorem 2 follows from /1/.

2. $r = 1$. It follows from Remark 1 that $n \geqslant k$. Suppose the assertion is false. Then for any $T > 0$ there exists a strategy V for player E such that for any trajectories $x_i(t)$ of players P_i we have (2.4).

We may assume that the vectors $z_1^\circ, \ldots, z_k^\circ$ form a basis of R^k.

Define counter-strategies U_i for players P_i as follows:

$$u_l^i(t) = v_l(t) - \rho_i(z_i^c / \|z_i^c\|, v_l(t))\, z_i^\circ / \|z_i^c\|$$

where $\sigma = \{0 = t_0 < t_1 < \ldots < t_{s+1} = T\}$, and

$$\rho_i(a_i, v) = \langle a_i v\rangle + \sqrt{\langle a_i, v\rangle^2 + 1 - \|v\|^2}$$

It can be shown that the counter-strategies U_i are admissible. Since V is an admissible strategy, it follows that

$$\int_0^t e^{-a\tau} \langle p_1, v(\tau)\rangle \, d\tau \leqslant \mu_0, \quad \text{where} \quad \mu_0 = -\langle p_1, y^c\rangle$$

(we may assume that $\mu_1 = 0$).

Since $p_1, z_1^\circ, \ldots, z_n^\circ$ form a positive basis, it follows by Theorem 1 that

$$\delta = \min_{\|v\| \leqslant 1} \max_l \rho_l(a_l, v) > 0$$
$$\rho_l(a_l, v) = \langle a_l, v\rangle + \sqrt{\langle a_l, v\rangle^2 + 1 - \|v\|^2}$$
$$\rho_{n+1}(a_{n+1}, v) = \langle p_1, v\rangle$$

Let $T_1(t), T_2(t)$ be two subsets of the interval $[0, t]$ such that

$$T_1(t) = \{\tau \mid \tau \in [0, t], \langle p_1, v(\tau)\rangle < \delta\}$$
$$T_2(t) = \{\tau \mid \tau \in [0, t], \langle p_1, v(\tau)\rangle \geqslant \delta\}$$

Then

$$\delta I_2 - I_1 \leqslant \mu_0$$
$$I_1 + I_2 = a^{-1}(1 - e^{-at})$$

where

$$I_{1,2} = \int_{T_{1,2}(t)} e^{-a\tau}\, d\tau$$

The last two relations imply that

$$I_1 \geqslant [\delta(1 - e^{-at}) - a\mu_0]/[a(1+\delta)]$$

It follows from the definition of U_i and from system (2.2) that

$$\| z_i(t) e^{-at} \| = \| z_i^\circ \| - \int_0^t e^{-a\tau} \rho_i(a_i, v(\tau))\, d\tau$$

Hence

$$\sum_{i=1}^n \| z_i(t) e^{-at} \| \leqslant \sum_{i=1}^n \| z_i^\circ \| - \int_{T_1(t)} e^{-a\tau} \max_i \rho_i(a_i, v(\tau))\, d\tau \leqslant$$
$$\sum_{i=1}^n \| z_i^\circ \| + \mu_0\delta \mid (1+\delta) - [\delta^2(1 - e^{-at})] \mid [a(1+\delta)]$$

Consider the function

$$f(t) = \sum_{i=1}^n \| z_i^\circ \| + \mu_0\delta \mid (1+\delta) - [\delta^2(1 - e^{-at})] \mid [(1+\delta) a]$$

We have

$$f(0) > 0, \quad \lim_{t\to\infty} f(t) < 0$$

Hence there exists $T_0 > 0$ such that

$$f(T_0) = 0 \tag{2.5}$$

It follows from (2.5) that at a time T_0, at the latest, one of the functions $z_i(t)$ vanishes. This contradiction shows that capture must occur in Γ_1, no later than a time T_0.

3. r arbitrary, $r > 1$. There are two possibilities:

a) There exists l such that the vectors $p_l, z_1^\circ, \ldots, z_n^\circ$ form a positive basis of R^k. Consider the set

$$D_1 = \{x \mid x \in R^k, \langle p_l, x\rangle \leqslant \mu_l\}$$

Since $D \subset D_1$, we see that capture occurs in Γ_1.

b) For no l do the vectors $p_l, z_1^\circ, \ldots, z_n^\circ$ form a positive basis. We shall construct a set $D_1 = \{x \mid x \in R^k, \langle p_0 x\rangle \leqslant \mu_0\}$ such that $D \subset D_1$, and the vectors $p_0, z_1^\circ, \ldots, z_n^\circ$ form a positive basis.

Since $p_1, \ldots, p_r, z_1^\circ, \ldots, z_n^\circ$ form a positive basis of R^k, there exist $\alpha_1 > 0, \ldots, \alpha_r > 0$, $\beta_1 > 0, \ldots, \beta_n > 0$ such that

$$0 = \alpha_1 p_1 + \ldots + \alpha_r p_r + \beta_1 z_1^\circ + \ldots + \beta_n z_n^\circ$$

As p_0 we take the vector

$$p_0 = \alpha_1 p_1 + \ldots + \alpha_r p_r$$

If $p_0 = 0$, the vectors z_i° form a positive basis.

Indeed, let $x \in R^k$. Since by assumption z_l°, $l = 1, \ldots, k$ constitute a basis of R^k, there exist γ_l, $l = 1, \ldots, k$, such that

$$x = \gamma_1 z_1^\circ + \ldots + \gamma_k z_k^\circ$$

It follows from (2.6) that

$$x = \gamma_1 z_1^\circ + \ldots + \gamma_k z_k^\circ + d(\beta_1 z_1^\circ + \ldots + \beta_n z_n^\circ)$$

Taking d sufficiently large, we obtain

$$x = \gamma_1^\circ z_1^\circ + \ldots + \gamma_n^\circ z_n^\circ$$

where $\gamma_i^\circ > 0$.

If $p_0 \neq 0$, analogous reasoning will show that the vectors p_0, z_i° form a positive basis.

Consider the set

$$D_1 = \{x \mid x \in R^k, \langle p_0, x\rangle \leqslant \mu_0\}$$

where $\mu_0 = \sum_{l=1}^r \alpha_l \mu_l$. Clearly, $D \subset D_1$. This completes the proof of Theorem 2.

3. Consider the game Γ. Define a function $f(n) = \min\{m \mid$ encounter avoidance occurs in Γ for any admissible $z^\circ\}$. Theorem 2 and the results of /4, 5/ imply

Theorem 3. Let D be an unbounded polyhedral set. Then there exist $c_1(D) > 0$, $c_2(D) > 0$ such that, for any $n \neq 1$,

$$c_1(D)\, n \ln n \leqslant f(n) \leqslant c_2(D)\, n \ln n$$

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ON AN INTEGRABLE CASE OF PERTURBED KEPLERIAN MOTION*

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A general solution of a differential vector equation of perturbed Keplerian motion is derived for the case when the position vector and perturbing acceleration vector are collinear. A variable change is employed, in which the new independent variable is expressed in terms of the initial values of the phase variables and time, using the elliptical Jacobi function. The two-point boundary value problem for the initial equation is reduced to the Cauchy problem. A parametric representation is obtained for the regularized trajectory of motion of a material point under the action of a central force.

Let us consider a differential vector equation of perturbed Keplerian motion

$$\mathbf{r}^{\cdot\cdot} = -\mu\mathbf{r}r^{-3} + w\mathbf{r} \tag{1}$$

in which $\mathbf{r}, \mathbf{r}^{\cdot\cdot}$ are the vectors of position and acceleration of a material point, μ is the gravitational constant of the centre of attraction and w is a constant.

The differential Eq.(1) determines the intermediate orbits of a geocentric satellite four-body problem /1/, and of the known geocentric planetary problem of n bodies /2/.

A general integral of the equation of the type (1) appears in a number of papers (e.g. in /3/), but is not solved for the required coordinates of the vector $\mathbf{r}(x, y, z)$.

We shall assume that the following initial conditions are specified in the initial coordinate system for the instant $t = t_0$:

$$\mathbf{r}(t_0) = (x_0, y_0, z_0), \quad \mathbf{r}^{\cdot}(t_0) = (x_0^{\cdot}, y_0^{\cdot}, z_0^{\cdot}).$$

Let us bring into our discussion the constant vector of angular momentum and the oscillating Laplace vector

$$\mathbf{h} = [\mathbf{r}, \mathbf{r}^{\cdot}] = (h_1, h_2, h_3), \quad \mathbf{l} = [r^{\cdot}, h] - \mu\mathbf{r}r^{-1}$$

The differential equation for l now takes the form

$$\mathbf{l}^{\cdot} = -\mu^{-1}wr\,[\mathbf{l}, \mathbf{h}]$$

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